Performance Certification of Interconnected Nonlinear Systems using ADMM

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Abstract

We present a compositional performance certification method for interconnected systems, using dissipativity properties of the subsystems along with the interconnection structure. To select the most relevant dissipativity properties, we formulate a large-scale optimization problem, and employ the alternating direction method of multipliers (ADMM) for its solution. The dissipativity properties are allowed to depend on an unknown equilibrium, enabling us to certify performance without explicit knowledge of the equilibrium for the interconnection. The effectiveness of the algorithm is demonstrated on two examples, including a model of vehicle platoons.

1 Introduction

In this paper, we consider compositional analysis for performance certification of an interconnection of nonlinear subsystems as depicted in Figure 1. The G_i are known subsystems mapping $u_i \mapsto y_i$, with dynamics described by nonlinear state equations. M is a static matrix that characterizes the interconnection topology. The overarching goal of compositional analysis is to establish properties of the interconnected system using only properties of the subsystems along with information about the interaction of the subsystems (M). Henceforth, the term "local" is used to refer to properties or analysis of individual subsystems in isolation. Likewise, "global" refers to the entire interconnected system.

In this paper, local and global properties are cast and quantified in the framework of dissipative systems [14]; specifically the case with quadratic supply rates [15]. The choice of a supply rate dictates the specific property that is to be verified. For example, different supply rates can be used to verify L_2-gain properties, passivity, output-strict passivity, etc., with respect to the exogenous input d and performance output e.

A conventional approach to compositional analysis, as presented in [1, 2, 10, 14] and others is to establish individual supply rates (and storage functions) for which each subsystem is dissipative. Then, a candidate storage function for the interconnected system is sought as a linear combination of the subsystem storage functions.

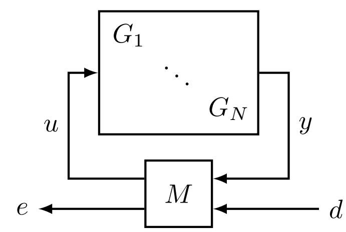

Figure 1: Interconnected system with exogenous input d and performance output e.

The method presented here and in [12] is less conservative in that the local storage functions are treated as decision variables, and are optimized with regards to their particular suitability in certifying desired properties of the specific interconnection defined by M. Thus, global properties are certified via local certificates that have been automatically generated, as opposed to having been preselected.

The idea of optimizing over the local supply rates (and storage functions) to certify stability of an interconnected system was introduced in [12], with the individual supply rates constrained to be diagonally-scaled induced L_2-norms. This perspective coupled with dual decomposition, gave rise to a distributed optimization towards certifying the stability of the interconnection and was solved using subgradient techniques. More recent work in [9] generalized this in several ways: certifying dissipativity (rather than stability) of the interconnected system with respect to a quadratic supply rate; using arbitrary quadratic supply rates for the local subsystems; and employing ADMM [3] which exposes the distributed certification as a convergent negotiation between local computational agents for each subsystem, and a master agent. The roles of the local and master agents are as follows:

- Local agents solve uncoupled, parallelizable problems for each subsystem. Beginning with a "proposed" supply rate provided by the master agent, the local agent uses the subsystem dynamical model to minimize the difference between the proposed supply-rate and a supply-rate for which the subsys-

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tem's dissipativity can be established.

- The master agent involves only supply-rates and the interconnection topology M (but no dynamical models), reassessing the supply-rates that should be proposed to each subsystem, based on the results of the previous iteration.

In this paper, we continue along the direction proposed in [9]. Unlike the linear system models in [9], we apply this method to nonlinear subsystems using SOS-optimization for subsystem dissipativity certification. Additionally, we use the concept of equilibrium-independent dissipativity for interconnections where the equilibrium point itself is unknown.

2 Preliminaries

Dissipativity theory [14]. Consider a time-invariant, continuous-time dynamical system described by

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \quad f(0,0) = 0 \\ y(t) &= h(x(t), u(t)), \quad h(0,0) = 0 \end{aligned} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. A *supply rate* is a function $w : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$. A system of the form (1) is *dissipative* with respect to a supply rate w if there exists a differentiable and nonnegative function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\nabla V(x)^{\mathsf{T}} f(x, u) - w(u, y) \le 0 \tag{2}$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y = h(x, u)$. Equation (2) is referred to as the *Dissipation Inequality Equation* (DIE) and V as a *storage function.*

Equilibrium independent dissipativity [EID]. EID was formalized in [7] and can be used to analyze systems where the equilibrium point depends nontrivially on the system input.

Consider a system of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned} \tag{3}$$

such that for every $x^* \in \mathbb{R}^n$ there exists a unique $u^* \in \mathbb{R}^m$ such that $f(x^*, u^*) = 0$. The *equilibrium state-input map* is then defined as

$$k_u(x) : \mathbb{R}^n \to \mathbb{R}^m \text{ such that } u^* = k_u(x^*) \tag{4}$$

and the *equilibrium state-output map* is defined as

$$\begin{aligned} &k_y(x) : \mathbb{R}^n \to \mathbb{R}^p \text{ such that} \\ &y^* = k_y(x^*) = h(x^*, k_u(x^*)) \end{aligned} \tag{5}$$

A system of the form (3) is *EID* with respect to the supply rate, w, if for every $x^* \in \mathbb{R}^n$ there exists a nonnegative storage function, $V : \mathbb{R}^{2n} \to \mathbb{R}_+$, such that $V(x^*, x^*) = 0$ and

$$\nabla_x V(x, x^*) f(x, u) - w(u - u^*, y - y^*) \le 0 \tag{6}$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ where $y = h(x, u)$ and $y^* = k_y(x^*)$.

Note that the definition given here is slightly different than in [7] and follows the convention in [4] and [5]. We assume the existence of an *equilibrium state-input* map whereas in [7] it is assumed that an *equilibrium input-state* map exists. This definition allows systems such as an integrator to be EID.

SOS programming. Let $\mathbb{R}[x]$ be the set of all polynomials in x with real coefficients. $\Sigma[x] := \{\pi \in \mathbb{R}[x] : \pi = \pi_1^2 + \pi_2^2 + \cdots + \pi_n^2, \quad \pi_1, \ldots, \pi_n \in \mathbb{R}[x]\}$ is the subset of $\mathbb{R}[x]$ containing the SOS polynomials. A polynomial, $p(x)$, being a sum of squares polynomial is equivalent to the existence of a positive semidefinite matrix Q such that

$$p(x) = m^{\top}(x) Q m(x) \tag{7}$$

for a properly chosen vector of monomials, $m(x)$. Therefore checking the nonnegativity of a polynomial can be relaxed to a SOS program and then solved as a semidefinite program (SDP).

Polynomial dynamics. Suppose that f and h in (1) are polynomials then certification of dissipativity of the system with respect to a given polynomial supply rate, w, can be relaxed to a SOS feasibility program:

$$\begin{aligned} &V(x) \in \Sigma[x] \\ &-\nabla V(x)^{\top} f(x, u) + w(u, y) \in \Sigma[x, u] \end{aligned} \tag{8}$$

Similarly, as presented in [7], certification of EID for polynomial systems can be relaxed to a SOS feasibility program:

$$\begin{aligned} &V(x, x^*) \in \Sigma[x, x^*] \\ &r(x, x^*, u, u^*) \in \mathbb{R}[x, u, x^*, u^*] \\ &-\nabla_x V(x, x^*)^{\top} f(x, u) + w(u - u^*, y - y^*) \\ &\qquad + r(x, x^*, u, u^*) f(x^*, u^*) \in \Sigma[x, u, x^*, u^*] \end{aligned} \tag{9}$$

Additionally, minimization of an objective subject to the constraints in (8) or (9) can also be performed. In this case since w enters (8) and (9) linearly it may be a decision variable.

Rational polynomial dynamics. If the system dynamics, f_i, are described by rational polynomials of the form

$$f_i(x, u) = \frac{p_i(x, u)}{q_i(x, u)} \quad \text{for all } i \in [1, \ldots, n] \tag{10}$$

where $p_i \in \mathbb{R}[x, u]$ and $q_i - \epsilon \in \Sigma[x, u]$ for some positive ϵ, then certifying dissipativity of the system with respect to a polynomial supply rate, w, can be relaxed to a SOS feasibility program:

$$\begin{aligned}
&V(x) \in \Sigma[x] \\
&-\sum_{i=1}^{n} \nabla_{x_i} V(x) p_i(x,u) \prod_{j\neq i} q_j(x,u) \\
&\qquad\qquad + \prod_{i=1}^{n} q_i(x,u) w(u,y) \in \Sigma[x,u]
\end{aligned} \tag{11}$$

As in the polynomial case this can be modified to certify EID or to allow minimization of an objective.

3 Problem statement

Consider the interconnected system in Figure 1 which consists of N known subsystems, G_i, with a known, static interconnection $M \in \mathbb{R}^{m\times p}$. Therefore,

$$\begin{bmatrix} u \\ e \end{bmatrix} = M \begin{bmatrix} y \\ d \end{bmatrix} \tag{12}$$

Each G_i has dynamics of the form (1) and is characterized by a local state x_i. We assume that the interconnected system is well-posed, meaning that for any $d \in L_{2e}$, and any initial condition x_0 there exists unique $e, u, y \in L_{2e}$ that causally depend on d. We assume that the subsystems are EID and there exists an equilibrium point x^* for which there is a unique $u^* = k_u(x^*)$, $y^* = k_y(x^*)$, and $e^* = M_{21} y^*$.

We assume that the global and local supply rates are quadratic forms. In particular, the given global supply rate is

$$\begin{bmatrix} d \\ e - e^* \end{bmatrix}^\mathsf{T} W \begin{bmatrix} d \\ e - e^* \end{bmatrix} \tag{13}$$

and the local supply rates are

$$\begin{bmatrix} u_i - u_i* \\ y_i - y_i* \end{bmatrix}^\mathsf{T} X_i \begin{bmatrix} u_i - u_i* \\ y_i - y_i* \end{bmatrix} \tag{14}$$

where W and all X_i are real symmetric matrices. To certify the desired global dissipativity from local EID properties we pose an optimization problem of the form

$$\begin{aligned}
\underset{X_{1:N}, V_{1:N}}{\text{minimize}} \quad & 0 \\
\text{subject to} \quad & X_i \in \mathcal{L}_i \quad \text{for } i = 1, \ldots, N \\
& (X_1, \ldots, X_N) \in \mathcal{G}
\end{aligned} \tag{15}$$

Each $\mathcal{L}_i$ constraint is *local* because it involves only the local supply rate X_i and associated storage function V_i. The $\mathcal{G}$ constraint is *global* because it involves all the supply rates.

Before we define the $\mathcal{L}_i$ and $\mathcal{G}$ sets, first introduce the following conformal block partitions

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \qquad X_i = \begin{bmatrix} X_i^{11} & X_i^{12} \\ X_i^{21} & X_i^{22} \end{bmatrix}$$

and the following block-diagonal matrices

$$X^{jk} = \begin{bmatrix} X_1^{jk} & & \\ & \ddots & \\ & & X_N^{jk} \end{bmatrix} \qquad \text{for all } j,k \in \{1,2\}$$

Recall that W is given while $X_1, \ldots, X_N$ are to be found. The local and global sets are defined as follows.

$$\begin{aligned}
\mathcal{L}_i := \Bigg\{ X_i \;\Bigg|\; &\text{the } i\text{-th subsystem is EID w.r.t} \\
&\text{the supply rate } \begin{bmatrix} u_i - u_i* \\ y_i - y_i^* \end{bmatrix}^\mathsf{T} X_i \begin{bmatrix} u_i - u_i^* \\ y_i - y_i^* \end{bmatrix} \Bigg\}
\end{aligned} \tag{16}$$

$$\mathcal{G} := \left\{ X_{1:N} \;\middle|\; \sum_{i=1}^{N} H_i X_i H_i^\mathsf{T} - H_0 W H_0^\mathsf{T} \preceq 0 \right\} \tag{17}$$

Here, the constant matrices $H_0, \ldots, H_N$ are defined such that the following identity holds.

$$\begin{aligned}
&\sum_{i=1}^{N} H_i X_i H_i^\mathsf{T} - H_0 W H_0^\mathsf{T} = \\
&\quad \begin{bmatrix} M \\ I \end{bmatrix}^\mathsf{T} \begin{bmatrix} X^{11} & 0 & X^{12} & 0 \\ 0 & -W_{22} & 0 & -W_{21} \\ X^{21} & 0 & X^{22} & 0 \\ 0 & -W_{12} & 0 & -W_{11} \end{bmatrix} \begin{bmatrix} M \\ I \end{bmatrix}
\end{aligned} \tag{18}$$

In [9] it was shown that if each subsystem is dissipative with respect to a supply rate, X_i, and that $(X_1, \ldots, X_N) \in \mathcal{G}$ then the interconnected system is dissipative with respect to the global supply rate. This result extends directly to the EID case: if the subsystems are EID and the global constraint is satisfied then the interconnected system is dissipative.

The benefit of the formulation in (15) is that verifying feasibility of a candidate point may be carried out in an efficient manner. The local constraints only depend on the storage function and supply rate for the associated subsystem, so they can be checked separately and in parallel. Finally, membership of the $\mathcal{G}$ set amounts to solving a global LMI. This constraint cannot be decoupled, but it does not depend on the storage functions. There are disadvantages as well. Any dissipativity property that can established for the interconnected system in this manner must be consistent with a storage function that is additively separable in the subsystem states. In general, the expressiveness of such storage functions are limited.

4 ADMM

In [9] we demonstrated that ADMM, described in detail in [3], could be used to decompose and solve (15) reliably and efficiently. This method is advantageous because it allows us to solve the individual subproblems, which require searching for a potentially high order storage function, separately.

ADMM can be used to solve problems of the form

$$\begin{array}{ll} \text{minimize} & f(x)+g(z) \\ \text{subject to} & Ax+Bz=c \end{array} \tag{19}$$

where x and z are vector decision variables. Our problem (15) may be put into this form by defining the following indicator functions:

$$\mathbb{I}_{\mathcal{L}_i}(X_i) := \begin{cases} 0 & X_i \in \mathcal{L}_i \\ \infty & \text{otherwise} \end{cases}$$
$$\mathbb{I}_{\mathcal{G}}(X_{1:N}) := \begin{cases} 0 & (X_1,\ldots,X_N) \in \mathcal{G} \\ \infty & \text{otherwise} \end{cases}$$

Then (15) may be written as

$$\begin{array}{ll} \underset{X_{1:N},Z_{1:N},V_{1:N}}{\text{minimize}} & \sum_{i=1}^{N} \mathbb{I}_{\mathcal{L}_i}(X_i) + \mathbb{I}_{\mathcal{G}}(Z_1,\ldots,Z_N) \\ \text{subject to} & X_i - Z_i = 0 \quad \text{for } i=1,\ldots,N \end{array}$$

so that it is in the canonical form (19). The f function and the constraints are separable for each subsystem, so the ADMM update takes on the following parallelized form [3].

1. X-updates: for each i, solve the local problem
$$X_i^{k+1} = \underset{X\in\mathcal{L}_i}{\arg\min} \left\| X - Z_i^k + U_i^k \right\|_F^2$$
2. Z-update: if $X_{1:N}^{k+1}$ is not feasible, solve the global problem
$$Z_{1:N}^{k+1} = \underset{Z_{1:N}\in\mathcal{G}}{\arg\min} \left\| \sum_{i=1}^{N} (X_i^{k+1} - Z_i + U_i^k) \right\|_F^2$$
3. U-update: perform the following update and return to step 1.
$$U_i^{k+1} = X_i^{k+1} - Z^{k+1} + U_i^k$$

When applied to (19), ADMM is guaranteed to converge if f, g are closed, proper, and convex, and the Lagrangian has a saddle point [3].

5 Examples

In this section we present two examples illustrating the proposed approach. The first example demonstrates the scalability and reliability of this approach for certifying performance of large interconnected systems with rational polynomial dynamics. The second example is based on the model of vehicle platoons used in [4] and [5].

A large-scale polynomial system The system consists of N, 2-state subsystems that share a common dynamical structure, modified from [8], but with different parameters defining each subsystem.

An individual subsystem, H, has 2 states and is described by

$$H: \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-ax_2 - bx_1^3 + u}{1+cx_2^2} \\ y &= x_2 \end{aligned} \tag{20}$$

where $a, b, c > 0$ are parameters of the subsystem. For purposes of understanding how the larger example is constructed, as well as the expectations of the example, we first analyze some input/output properties of this system.

The positive-definite storage function $V(x) := \frac{ab}{2}x_1^4 + \frac{ac}{2}x_2^4 + ax_2^2$ certifies that the L_2-gain of H is bounded by a^{-1}. Indeed, define $w(u,y) := u^2 - a^2y^2$, and check that

$$\nabla V(x) f(x,u) - w(u,y) = -(ax_2 - u)^2 \le 0$$

Clearly, *any* well-posed interconnection involving many instances of these systems, along with an interconnection matrix whose spectral norm is less than a results in a dynamical systems whose L_2 gain is less than a. Armed with this insight, we can construct large-scale examples for which the SOS/ADMM certification procedure should succeed.

The steps are:

1. Choose $\{a_i, b_i, c_i\}_{i=1}^N$ uniformly distributed in $(1,2)\times(0,1)\times(0.5,2)$. These constitute the parameters of system H_i. Denote $\gamma := \max_i a_i^{-1}$.
2. Choose $S \in \mathbb{R}^{(N+d)\times(N+d)}$ using a normal distribution. Overlay any desired sparsity pattern by selectively zeroing out particular entries.
3. Compute $\beta := \inf_B \bar{\sigma}(BMB^{-1})$ where $B = \mathrm{diag}(b_1,\ldots,b_N,I_d)$, $b\in\mathbb{R}_{++}^N$. Redefine $S := \frac{0.99}{\gamma\beta}S$.
4. Choose random nonzero, diagonal scalings $\Psi = \mathrm{diag}(\Psi_1,\ldots,\Psi_N)$ and $\Phi = \mathrm{diag}(\Phi_1,\ldots,\Phi_N)$.
5. Define $G_i := \Phi_i H_i \Psi_i$, and
$$M := \begin{bmatrix} \Psi^{-1} & 0 \\ 0 & I_d \end{bmatrix} S \begin{bmatrix} \Phi^{-1} & 0 \\ 0 & I_d \end{bmatrix}$$
6. Apply SOS/ADMM algorithm to the data $\{G_i\}_{i=1}^N$ and M, attempting to certify the L_2-gain from $d \to e$ is less than γ.

Figure 2 below illustrates the interconnection (whose simplicity is now masked by the scalings) that the algorithm must attempt to certify.

For testing purposes, we generated 100 random instances of the interconnected system described above,

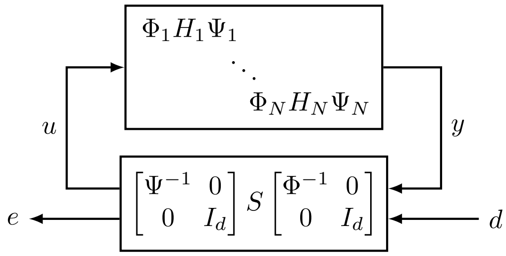

Figure 2: Scaling of the interconnected system of Figure 1 that leaves the closed-loop map unchanged.

each with $N = 100$. The SOS/ADMM algorithm was applied to these instances, attempting to certify the L_2-gain of the interconnected system is less than or equal to γ. The storage functions within each local SOS dissipativity certification were restricted to be quartic polynomials.

Figure 3 shows the result of this test. The algorithm succeeded for all 100 tests, requiring at most 48 iterations. However, 90% of the tests succeeded in fewer than 15 iterations.

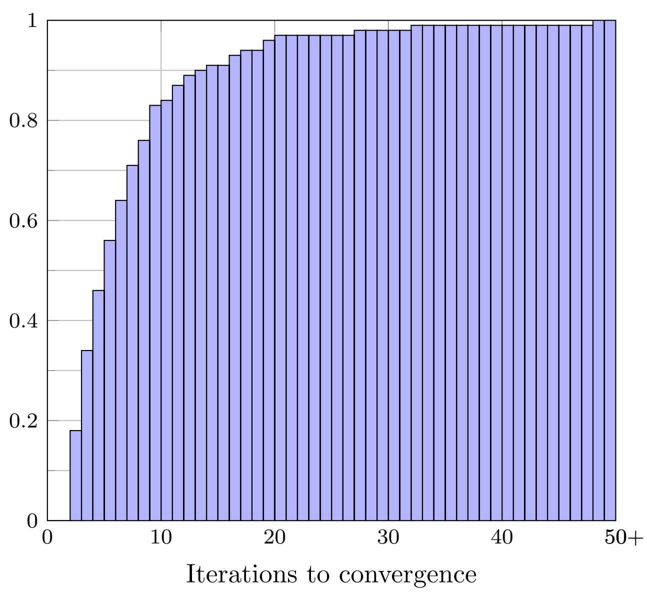

Figure 3: Cumulative plot displaying the fraction of 100 total tests that required at most a given number of iterations to certify the L_2-gain property of the interconnected system. The fastest trials succeeded in 3 iterations and 90% succeeded in fewer than 15 iterations.

Vehicle platooning. In this example we analyze the L_2-gain properties of a model for a vehicle platoon. A possible configuration is shown in Figure 4. Each vehicle measures its distance to a subset of vehicles (indicated by dashed lines), and adjusts its throttle according to some control law. We would like to certify that under a broad range of control law choices and measurement topologies, bounds on the L_2-gain properties can be certified. For the purpose of this illustration, we ignore vehicle collisions.

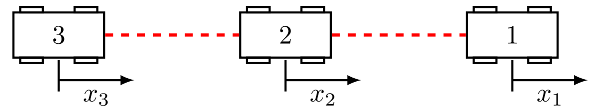

Figure 4: Vehicle platoon example. Each vehicle measures the relative distance of all vehicles connected to it by a dotted line. The objective is that all vehicles eventually move at a common speed.

Control strategies for velocity regulation were presented and analyzed for a similar scenario in [4, 5]. We will consider a general set of control strategies that encompasses those presented in [4, 5].

Consider N vehicles in a platoon. The dynamics of the i^{th} vehicle are described by

$$\Sigma_i : \quad \begin{aligned} \dot{v}_i &= -v_i + v_i^{\text{nom}} + u_i \\ y_i &= v_i \end{aligned} \qquad i = 1, \ldots, N$$

where $v_i(t)$ is the vehicle velocity and v_i^{nom} is a nominal velocity. In the absence of a control input $u_i(t)$, each vehicle eventually tends to its nominal velocity.

Each vehicle uses the relative distance between itself and a subset of the other vehicles to control its velocity. The subsets are represented by a bidirectional graph with L links interconnecting the N vehicles. In Figure 4, the links are shown as dotted lines. Letting p_ℓ be the relative displacement between the vehicles connected by link ℓ then $\dot{p}_\ell = v_i - v_j$ where x_i is the leading node and x_j is the trailing node. We define $D \in \mathbb{R}^{N \times L}$ as

$$D_{i\ell} = \begin{cases} 1 & \text{if } i \text{ is the leading node of edge } \ell \\ -1 & \text{if } i \text{ is the trailing node of edge } \ell \\ 0 & \text{otherwise} \end{cases}$$

Thus, D maps the individual velocities of the vehicles to the relative velocities across links. That is, $\dot{p} = D^\top v$. We will analyze control laws of the form

$$u_i = -\sum_{\ell=1}^{L} D_{i\ell} \phi_\ell(p_\ell)$$

where $\phi_\ell : \mathbb{R} \to \mathbb{R}$ is increasing and surjective, which ensures the existence of an equilibrium point [4]. Defining $\Phi := \operatorname{diag}(\phi_1, \ldots, \phi_L)$, we may represent the system as in the block diagram of Figure 5.

The map Λ from $\dot{p}$ to $\Phi(p)$, indicated by a dashed box in Figure 5, is diagonal; each $\dot{p}_\ell$ is separately integrated and then the corresponding ϕ_ℓ is applied. Thus, we may write $\Lambda = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_L)$, where Λ_ℓ is described by

$$\Lambda_\ell : \quad \begin{aligned} \dot{p}_\ell &= \eta_\ell \\ z_\ell &= \phi_\ell(p_\ell) \end{aligned} \qquad \ell = 1, \ldots, L$$

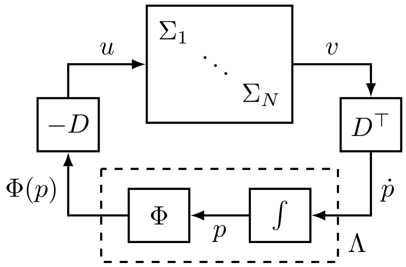

Figure 5: Block diagram representation of vehicle platoon dynamics.

where η_ℓ is the input and z_ℓ is the output. The interconnection structure may be transformed into the standard form of Figure 1 by diagonally concatenating $\Sigma := \mathrm{diag}(\Sigma_1, \ldots, \Sigma_N)$. The resulting block diagram is shown in Figure 6.

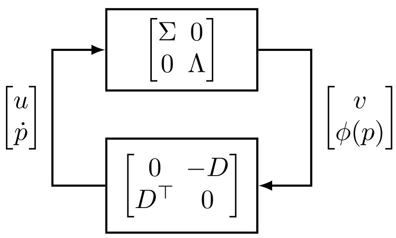

Figure 6: Transformation of Figure 5 into the standard form shown in Figure 1.

Although an equilibrium (v^*, p^*) exists its location depends on the nonlinearities ϕ_ℓ and is difficult to compute. Instead we will exploit the EID properties of the subsystems and establish the desired global property without explicit knowledge of the equilibrium.

For each Σ_i subsystem, we may certify EID by formulating and solving a SOS program. In this case, the Σ_i dynamics are linear, so the SOS program reduces to a simpler LMI.

For each Λ_ℓ subsystem, the SOS program depends on the choice of ϕ_ℓ functions. However, it is not difficult to show that Λ_i is EID with respect to the following supply rate (also known as *equilibrium independent passivity (EIP)*)

$$\begin{bmatrix}\eta_\ell - \eta_\ell^*\\ z_\ell - z_\ell^*\end{bmatrix}^\top \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \begin{bmatrix}\eta_\ell - \eta_\ell^*\\ z_\ell - z_\ell^*\end{bmatrix} \tag{21}$$

This property can be proven by using the storage function

$$V_\ell(p_\ell) = 2\int_{p_\ell^*}^{p_\ell} [\phi_\ell(\theta) - \phi_\ell(p_\ell^*)]\, d\theta \tag{22}$$

and the property $(p_\ell - p_\ell^*)[\phi_\ell(p_\ell) - \phi_\ell(p_\ell^*)] \geq 0$ which follows because ϕ_ℓ is increasing. Therefore, instead of searching over supply rates for the Λ_ℓ subsystems in the ADMM algorithm, we fix (21) as the supply rate. Note that even though the ϕ_ℓ functions and their associated equilibrium points are unknown, the ADMM algorithm does not require them; it is enough to know the supply rate (21).

For the simulation, we used $N = 20$, and each vehicle's nominal velocity was randomly generated. A linear topology was used as in Figure 4. That is, each vehicle measures the distance to the vehicle in front of it and the vehicle behind it. We investigated how a force disturbance applied to the trailing vehicle would affect the velocity of the lead vehicle. Specifically, we augmented the interconnection matrix M (see Figure 1) such that the disturbance d is applied to the last vehicle:

$$\dot{v}_N = -v_N + v_N^{\mathrm{nom}} + u_N + d$$

and the output e is the velocity of the first vehicle v_1. We then attempted to certify EID of the interconnected system using a supply rate of the form

$$\begin{bmatrix}d\\ e - e^*\end{bmatrix}^\top \begin{bmatrix}1 & 0\\ 0 & -\gamma^{-2}\end{bmatrix} \begin{bmatrix}d\\ e - e^*\end{bmatrix}$$

Successful certification implies that the L_2 gain from d to e is no greater than γ. Using a bisection search to find the least certifiable γ, we found that $\gamma_{\min} \leq 0.71$. The inequality is due to the fact that our method searches over a restricted class of possible storage functions, and may therefore be conservative. In an effort to bound this conservatism, we performed an ad-hoc search over linear ϕ_ℓ functions, seeking a worst-case L_2 gain. The result was that $\gamma_{\min} \geq 0.49$.

Certain provisions were made to achieve satisfactory convergence of the ADMM routine, and this highlights a potential pitfall of our method that is common to many iterative optimization algorithms. For this platoon example, the local subproblems have among them the constraints $X_i^{11} \geq 0$ for $i = 1, \ldots, N$ (these are scalar variables), while the global problem has the constraint $D^\top \mathrm{diag}(X_1^{11}, \ldots, X_N^{11}) D \leq 0$. One can check that the only solution is $X_i^{11} = 0$ for all i. This further implies that $X_i^{12} = X_i^{21} = 1$ for all i. Intuitively, $\mathcal{G}$ as well as the $\mathcal{L}_i$ sets defined in (15) each have a nonempty interior but their intersection does not. The result is that ADMM oscillates between $X_i^{11} > 0$ for the local problems and $Z_i^{11} < 0$ for the global problem, leading to slow convergence and only reaching feasibility in the limit. We addressed this issue by setting $X_i^{11} = 0$ and $X_i^{12} = X_i^{21} = 1$, effectively removing those variables from the optimization. The feasible region of the resulting problem has a nonempty interior, and our ADMM algorithm converged in a few iterations.

An alternative approach that does not rely on exploiting the structure of the local and global constraints was also used successfully. After, 30 iterations the global constraint matrix was close to negative semidefinite (i.e. the maximum eigenvalue was less than 10^{-5}). By examining the supply rates found by the algorithm for the Σ_i subsystems it was apparent that the supply rates found by the global and local optimization steps, as can be seen in Figure 7, were approaching the form

$$\begin{bmatrix} u_i \\ y_i \end{bmatrix}^\top \begin{bmatrix} 0 & 1 \\ 1 & c_i \end{bmatrix} \begin{bmatrix} u_i \\ y_i \end{bmatrix} \tag{23}$$

where c_i differed for each subsystem.

In light of this observation, the supply rates were set to (23) where c_i was unchanged from the value determined by the algorithm. These supply rates were found to satisfy the local and global constraints resulting in the same L_2 gain bound.

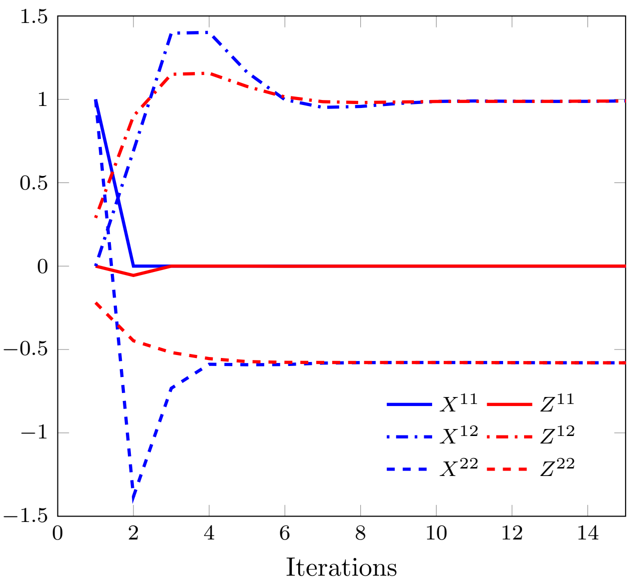

Figure 7: Convergence of elements of the supply rate matrices.

All simulations performed were implemented in MATLAB using the SOSOPT toolbox [11] to solve SOS programs and the CVX toolbox [6] to solve SDP problems.

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References

[1] J. Anderson, A. Teixeira, H. Sandberg, and A. Papachristdoulou. Dynamical system decomposition using dissipation inequalities. In *IEEE Conference on Decision and Control*, pages 211–216, 2011.

[2] M. Arcak and E. D. Sontag. A passivity-based stability criterion for a class of biochemical reaction networks. *Mathematical Biosciences and Engineeering*, 5(1):1–19, 2008.

[3] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends in Machine Learning*, 3(1):1–122, 2011.

[4] M. Bürger, D. Zelazo, and F. Allgöwer. Duality and network theory in passivity-based cooperative control. In *arXiv preprint arXiv:1301.3676*, 2013.

[5] S. Coogan and M. Arcak. A dissipativity approach to safety verification for interconnected systems. In *IEEE Transactions on Automatic Control, Submitted*, 2014.

[6] M. Grant and S. Boyd. CVX: Matlab software for disciplined convex programming, version 2.0 beta. `http://cvxr.com/cvx`, Sept. 2013.

[7] G. Hines, M. Arcak, and A. Packard. Equilibrium-independent passivity: a new defintion and numerical certification. In *Automatica*, pages 1949–1956, 2011.

[8] H. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd edition, 2002.

[9] C. Meissen, L. Lessard, and A. Packard. Performance certification of interconnected systems using decomposition techniques. In *American Control Conference, Accepted*, 2014.

[10] N. Sandell, Jr., P. Varaiya, M. Athans, and M. Safonov. Survey of decentralized control methods for large scale systems. *IEEE Transactions on Automatic Control*, 23(2):108–128, 1978.

[11] P. Seiler. Sosopt: A toolbox for polynomial optimization. In *arXiv preprint arXiv:1308.1889*, 2013.

[12] U. Topcu, A. Packard, and R. M. Murray. Compositional stability analysis based on dual decomposition. In *IEEE Conference on Decision and Control*, pages 1175–1180, 2009.

[13] T. Ufuk. *Quantitative Local Analysis of Nonlinear Systems*. PhD thesis, University of California - Berkeley, 2008.

[14] J. C. Willems. Dissipative dynamical systems part I: General theory. *Archive for Rational Mechanics and Analysis*, 45(5):321–351, 1972.

[15] J. C. Willems. Dissipative dynamical systems part II: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45(5):352–393, 1972.